

Semi periodic maps on complex manifolds

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In this letter we proved this theorem: *if F be a holomorphic mapping of T_Ω to a mapping manifold X such that for every compact subset $K \subset \Omega$ the mapping F is uniformly continues on T_K and $F(T_K)$ is a relatively compact subset of X . If the restriction of $F(z)$ to some hyperplane $\mathbb{R}^m + iy'$ is semi periodic, then $F(z)$ is an semi mapping of T_Ω to X .*

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HISTORY

In mathematics, an almost(semi) periodic function is, loosely speaking, a function of a real number that is periodic to within any desired level of accuracy, given suitably long, well-distributed "almost-periods". The concept was first studied by Harald Bohr and later generalized by Vyacheslav Stepanov, Hermann Weyl and Abram Samoilovitch Besicovitch, amongst others. There is also a notion of almost periodic functions on locally compact abelian groups, first studied by John von Neumann.

Almost periodicity is a property of dynamical systems that appear to retrace their paths through phase space, but not exactly. An example would be a planetary system, with planets in orbits moving with periods that are not commensurable (i.e., with a period vector that is not proportional to a vector of integers). A theorem of Kronecker from diophantine approximation can be used to show that any particular configuration that occurs once, will recur to within any specified accuracy: if we wait long enough we can observe the planets all return to within a second of arc to the positions they once were in. There are several different inequivalent definitions of almost periodic functions. An almost periodic function is a complex-valued function of a real variable that has the properties expected of a function on a phase space describing the time evolution of such a system. There have in fact been a number of definitions given, beginning with that of Harald Bohr. His interest was initially in finite Dirichlet series. In fact by truncating the series for the Riemann zeta function $\zeta(s)$ to make it finite, one gets finite sums of terms of the type

$$e^{(\sigma+it) \log n} \quad (1)$$

with s written as $\sigma + it$ the sum of its real part σ and imaginary part it . Fixing σ so restricting attention to a single vertical line in the complex plane, we can see this also as

$$n^\sigma e^{(\log n)it}. \quad (2)$$

Taking a finite sum of such terms avoids difficulties of analytic continuation to the region $\sigma < 1$. Here the 'frequencies' $\log n$ will not all be commensurable (they are as linearly independent over the rational numbers as the integers n are multiplicatively independent which comes down to their prime factorizations).

With this initial motivation to consider types of trigonometric polynomial with independent frequencies, mathematical analysis was applied to discuss the closure of this set of basic functions, in various norms.

The theory was developed using other norms by Besicovitch, Stepanov, Weyl, von Neumann, Turing, Bochner and others in the 1920s and 1930s.

DEFINITIONS AND SOME THEOREMS

A continuous mapping F of a tube

$$T_K = \{z = x + iy : x \in \mathbb{R}^m, y \in K \subset \mathbb{R}^m\} \quad (3)$$

to a metric space X is semi periodic if the family $F(z+t)_{t \in \mathbb{R}^m}$ of shifts along \mathbb{R}^m is a relatively compact set with respect to the topology of the uniform convergence of T_K .

Further, let X be a manifold, and F be a holomorphic mapping of a tube

$$T_\Omega = \{z = x + iy : x \in \mathbb{R}^m, y \in \Omega\} \quad (4)$$

with the convex open base $\Omega \subset \mathbb{R}^m$, to X . We will say that F is semi periodic if the restriction of F to each tube T_K , with the compact base $K \subset \Omega$ is semi periodic.

For $X = \mathbb{C}$ we obtain the well-known class of holomorphic semi periodic functions; for $X = \mathbb{C}^q$ the corresponding class was being studied in [1–3]; for $X = \mathbb{C}P$ we get the class of meromorphic semi periodic functions that was being studied in [4]; the class of holomorphic semi periodic curves, corresponding to the case $X = \mathbb{C}P^q$, was being studied in [5].

Uniform or Bohr or Bochner almost periodic functions

The following theorem is due to Bohr[7]:

: Bohr[7] defined the uniformly almost-periodic functions as the closure of the trigonometric polynomials with respect to the uniform norm

$$\|f\|_\infty = \sup_x |f(x)| \quad (5)$$

(on continuous functions f on \mathbb{R}). He proved that this definition was equivalent to the existence of a relatively-dense set of ϵ almost-periods, for all $\epsilon > 0$: that is, translations $T(\epsilon) = T$ of the variable t making

$$|f(t+T) - f(t)| < \epsilon. \quad (6)$$

An alternative definition due to Bochner (1926) is equivalent to that of Bohr and is relatively simple to state:

A function f is almost periodic if every sequence $(t_n + T)$ of translations of f has a subsequence that converges uniformly for T in $(-\infty, \infty)$.

The Bohr almost periodic functions are essentially the same as continuous functions on the Bohr compactification of the reals. Stepanov almost periodic functions

The space Sp of Stepanov almost periodic functions was introduced by V.V. Stepanov [6]. It contains the space of Bohr almost periodic functions. It is the closure of the trigonometric polynomials under the norm

$$\|f\|_{S,r,p} = \sup_x \left(\frac{1}{r} \int_x^{x+r} |f(s)|^p ds \right)^{1/p} \quad (7)$$

for any fixed positive value of r ; for different values of r these norms give the same topology and so the same space of almost periodic functions (though the norm on this space depends on the choice of r). Weyl almost periodic functions

The space Wp of Weyl almost periodic functions was introduced by Weyl. It contains the space Sp of Stepanov almost periodic functions. It is the closure of the trigonometric polynomials under the seminorm

$$\|f\|_{W,p} = \lim_{r \rightarrow \infty} \|f\|_{S,r,p} \quad (8)$$

Warning: there are nonzero functions with $\|f\|_{W,p} = 0$, such as any bounded function of compact support, so to get a Banach space one has to quotient out by these functions. Besicovitch almost periodic functions

The space Bp of Besicovitch almost periodic functions was introduced by Besicovitch (1926). It is the closure of the trigonometric polynomials under the seminorm

$$\|f\|_{B,p} = \limsup_{x \rightarrow \infty} \left(\frac{1}{2x} \int_{-x}^x |f(s)|^p ds \right)^{1/p} \quad (9)$$

Warning: there are nonzero functions with $\|f\|_{B,p} = 0$, such as any bounded function of compact support, so to get a Banach space one has to quotient out by these functions.

The Besicovitch almost periodic functions in B2 have an expansion (not necessarily convergent) as

$$\sum a_n e^{i\lambda_n} \quad (10)$$

Conversely every such series is the expansion of some Besicovitch periodic function (which is not unique).

The space B_p of Besicovitch almost periodic functions

contains the space W_p of Weyl [9] almost periodic functions. If one quotients out a subspace of "null" functions, it can be identified with the space of L_p functions on the Bohr compactification of the reals. Almost periodic functions on a locally compact abelian group

With these theoretical developments and the advent of abstract methods (the PeterWeyl theorem, Pontryagin duality and Banach algebras) a general theory became possible. The general idea of almost-periodicity in relation to a locally compact abelian group G becomes that of a function F in $L^*(G)$, such that its translates by G form a relatively compact set. Equivalently, the space of almost periodic functions is the norm closure of the finite linear combinations of characters of G . If G is compact the almost periodic functions are the same as the continuous functions.

The Bohr compactification of G is the compact abelian group of all possibly discontinuous characters of the dual group of G , and is a compact group containing G as a dense subgroup. The space of uniform almost periodic functions on G can be identified with the space of all continuous functions on the Bohr compactification of G . More generally the Bohr compactification can be defined for any topological group G , and the spaces of continuous or L_p functions on the Bohr compactification can be considered as almost periodic functions on G . For locally compact connected groups G the map from G to its Bohr compactification is injective if and only if G is a central extension of a compact group, or equivalently the product of a compact group and a finite-dimensional vector space.

If a holomorphic bounded function on a strip is semi periodic on some straight line in this strip, then this function is semi periodic on the whole strip.

This theorem was extended to holomorphic functions on a tube in [8]; besides usual uniform metric, various integral metric were being studied here.

The direct generalization of Bohr's theorem to complex manifold is not valid.

THEOREM

Theorem: *if F be a holomorphic mapping of T_Ω to a mapping manifold X such that for every compact subset $K \subset \Omega$ the mapping F is uniformly continues on T_K and $F(T_K)$ is a relatively compact subset of X . If the restriction of $F(z)$ to some hyperplane $\mathbb{R}^m + iy'$ is semi periodic, then $F(z)$ is an semi mapping of T_Ω to X .*

Also there is a corollary:

Corollary: *Let F be a holomorphic mapping form T_Ω to a compact complex manifold X such that F is uniformly continues on T_K for every compact set $K \subset \Omega$. If the restriction of $F(z)$ to some hyperplane $\mathbb{R}^m + iy'$ is semi periodic, then $F(z)$ is an semi periodic mapping of T_Ω to X .*

Proof of the theorem:

Take an arbitrary sequence $\{t_n\} \subset \mathbb{R}^m$. Since the function $F(z)$ is uniformly continues, the family $\{F(z + t_n)\}$ is equicontinuous on each compact set $S \subset T_\Omega$. Further, it follows from the condition of the Theorem that the union of all the images of S under mappings of this family is contained compact subset of X . Therefore, passing on to a subsequence if necessary, we may assume that the sequence $\{F(z + t_n)\}$ converges to a holomorphic mapping $G(z)$ uniformly on every compact subset of T_Ω . It easy to see that the mapping $G(z)$ is bounded and uniformly continues on every tube T_K with the compact base $K \subset \Omega$. Let us prove that this convergence is uniform on every T_K . Assume the contrary. Then we get

$$||F(z + t_n), G(z_n)|| > 0 \quad (11)$$

for some sequence $z_n = x_n + iy_n \in T_{K'}$, where K' is some compact subset of Ω .

Replacing sequences by a subsequence if necessary, we may assume that the mapping $G(x_n + z)$ converge to a holomorphic mapping $H(z)$, and the mappings $F(z + x_n + t_n)$ converge to a holomorphic mapping $\tilde{H}(z)$ uniformly on every compact subsets of T_Ω . We may also assume that $y_n \rightarrow y_0 \in K'$. Using (1) we get

$$|\tilde{H}(iy_0) - H(iy_0)| \geq \varepsilon_0 \quad (12)$$

Since the mapping $F(x + iy')$ of \mathbb{R}^m to X is semi periodic, we may assume that a subsequence of mappings $F(x + t_n + iy')$ converges to $G(x + iy')$ uniformly in $x \in \mathbb{R}^m$. Therefor the sequences of mappings $F(x + x_n + t_n + iy')$ and $G(x + x_n + iy')$ have the same limit, i.e., $\tilde{H}(x + iy') = H(x + iy')$ for all $x \in \mathbb{R}^m$. Since $\tilde{H}(z), H(z)$ are holomorphic mappings we get $\tilde{H}(x + iy') = H(x + iy')$ on T_{Omega} . This contradiction proves the Theorem.

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